B.A./B.Sc. 5th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH5DSE21

(Probability and Statistics)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Show that the standard deviation is independent of any change of origin but dependent on change of scale.

(b) Find the value of the constant k such that $f(x) = \begin{cases} kx^2(1-x) & \text{if } 0 < x < 1 \\ 0 & elsewhere \end{cases}$

is a probability density function.

(c) Find the moment generating function of Binomial distribution.

(d) Prove that $E(X^2) \geq \{E(X)\}^2$.

(e) Define random variable and distribution function.

(f) Let X be a random variable with $E(X) = 1, E[X(X-1)] = 4$. Find variance of $(7-2X)$.

(g) Write down the mean and variance of standard normal distribution.

(h) If X_n is a binomial (n, p) variate, then show that $\frac{X_n}{n} \underset{in\,p}{\longrightarrow} p$ as $n \to \infty$.

(i) Define moment generating function for a continuous random variable.

(j) Let X be a random variable. Show that the distribution function $F(x)$ of X is a monotonic non-decreasing function.

(k) State the Central limit theorem.

(l) Define Type I and Type II errors.

(m) Two random variables X and Y have zero means and standard deviations 1 and 2 respectively. Find the variance of $X+Y$ if X and Y are uncorrelated.

(n) Let the random variable X have a normal distribution. Does the random variable $Y = X^2$ also have a normal distribution?

(o) For a normal (m, σ) population, prove that $(\bar{X} - m)/\frac{\sigma}{\sqrt{n}}$ is normal (0, 1) variate.

2. Answer *any four* questions: 5×4=20

(a) If X is uniformly distributed in the interval (–1, 1), then find the distribution of $|X|$.

(b) If X is a Poisson variate with parameter μ, then show that $P(X \leq n) = \frac{1}{n!}\int_{\mu}^{\infty} e^{-x}x^n dx$, where n is any positive integer.

(c) If the correlation coefficient $\rho(X, Y)$ between two variables X and Y exists, then show that $-1 \leq \rho(X, Y) \leq 1$.

(d) A random variable X has the probability mass function $f(x) = \frac{1}{2^x}, x = 1, 2, 3, \ldots$

Find its (i) moment generating function, (ii) mean and (iii) variance.

(e) What is sampling distribution of a statistic? Show that the function $t = \frac{\sqrt{\mu}(X - m)}{s}$ is a t-distribution with $n - 1$ degrees of freedom. 1+4

(f) Prove that the maximum likelihood estimate of the parameter α of the population having density function $f(x) = \frac{2(\alpha - x)}{\alpha^2}, (0 < x < \alpha)$, for a sample x_1 of unit size is $2x_1$ and this estimate is biased.

3. Answer *any two* questions: 10×2=20

(a) (i) If X and Y are two independent normal variates (m_X, σ_X) and (m_Y, σ_Y) respectively, then show that $U = X + Y$ is a normal variate (m, σ) where $m = m_X + m_Y$ and $\sigma^2 = \sigma_X^2 + \sigma_Y^2$.

(ii) If the random variable X is uniformly distributed over $(-2, 2)$, then find the mean and variance of the random variable $\min\{X, 1\}$. 5+5

(b) The joint probability density function of two random variables X and Y is $k(1 - x - y)$ inside the triangle formed by the axes and the line $x + y = 1$ and zero elsewhere. Find the value of k and $P\left(X < \frac{1}{2}, Y > \frac{1}{4}\right)$. Find also the marginal distributions of X, Y and determine whether the random variables are independent or not. 2+2+2+2+2

(c) (i) State Tchebycheff's inequality and weak law of large numbers.

(ii) The distribution of a random variable X is given by $P(X = -1) = \frac{1}{8}$, $P(X = 0) = \frac{3}{4}$, $P(X = 1) = \frac{1}{8}$. Verify Tchebycheff's inequality for the distribution. 2+2+6

(d) (i) Find the expectation of the sum of points on n unbiased dice.

(ii) If $X_1, X_2, \dots X_n, \dots$ be a sequence of random variables such that $S_n = X_1 + X_2 + \cdots X_n$ has a finite mean M_n and standard deviation Σ_n for all n. If $\frac{\Sigma_n}{n} \to 0$ as $n \to \infty$, then show that $\frac{S_n - M_n}{n} \xrightarrow[in\,P]{} 0$ as $n \to \infty$.

(iii) Prove that sample mean is unbiased estimate of the corresponding population mean provided the population mean exists.

4+3+3